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Non-relativistic superstrings: a new soluble sector of $\mathrm{AdS}_5 \times \mathrm{S}^5$

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Abstract: We find a new sector of string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ describing non-relativistic superstrings in that geometry. The worldsheet theory of non-relativistic strings in $\mathrm{AdS}_5\times\mathrm{S}^5$ is derived and shown to reduce to a supersymmetric free field theory in AdS_2. Non-relativistic string theory provides a new calculable setting in which to study holography.

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1. Introduction

Recent progress in string theory suggests that in order to define the theory non-perturbatively, we must specify the symmetries at asymptotic infinity, where observables can be defined. Examples of geometries where holographic duals can be constructed are asymptotically Anti-de Sitter and asymptotically linear dilaton backgrounds. The holographic field theory description encodes the complete non-perturbative string theory dynamics in these backgrounds. In practice, however, even reproducing perturbative string physics is hampered by the difficulties in solving the worldsheet theory.

Significant progress can be made in solving string theory in various backgrounds by considering a sector of the theory that decouples from the rest of the degrees of freedom in a suitable limit. Such decoupled sectors are characterized by having an altogether different asymptotic symmetry compared to that of the parent string theory. A well known example of such a truncation is the BMN [1] sector[1] of string theory in $AdS_5\times S^5$. Once a consistent sector is found, a complete worldsheet theory with the appropriate symmetries can be written down without further reference of the parent theory. Solving for the spectrum and interactions in the BMN sector has led to a concrete understanding of how perturbative string theory is encoded in the dual field theory and has shed new light on holography in the stringy regime.

Non-relativistic string theory [3] (see also [4]) in flat space is another example of a consistent decoupled sector of bosonic string theory whose worldsheet conformal field theory description [3] possesses the appropriate galilean symmetry. Non-relativistic string theory can be derived as a certain decoupling limit of the original relativistic theory, even though — as in the BMN case — the theory can be written down without further reference to the original parent theory.[2] The basic idea behind the decoupling limit is to take a particular *non-relativistic* limit in such a way that only states satisfying a galilean invariant dispersion relation have finite energy, while the rest decouple. This can be accomplished by considering wound strings in the presence of a background B-field and tuning the B-field so that the energy coming from the background B-field cancels the tension of the string. The corresponding worldsheet conformal field theory captures, in the usual fashion, the spectrum and interactions of these states. The computation of interactions among non-relativistic strings, however, simplifies considerably compared to the relativistic case, since the path integral of the worldsheet theory "localizes" [3] to points in the moduli space where holomorphic maps from the worldsheet to the target space exist. An analogous non-relativistic limit can be taken in the various corners of M-theory such that the non-relativistic states of M-theory are mapped to each other by the action of the duality symmetries of M-theory; thus giving rise to a rich duality web but in the non-relativistic setting.

In this paper we consider non-relativistic superstring[3] theory in $AdS_5\times S^5$. The primary motivation for studying this new limit of string theory is to isolate a simple sector of the AdS/CFT correspondence which is amenable to exact analysis. This should provide an interesting new arena in which to approach in a calculable setting the ideas of holography with $\mathcal{N}=4$ SYM. The main result of the paper is in fact to show that non-relativistic string theory in $AdS_5\times S^5$ is described by a supersymmetric two-dimensional sigma model of free massless and massive bosons and free massive fermions propagating in AdS_2, so that this sector is described by a *free* theory! We show that the non-relativistic string action in $AdS_5\times S^5$ has the supersymmetric Newton-Hooke group [14] symmetry, which is a supersymmetrization of the kinematical group describing non-relativistic particles in AdS. We make preliminary comments about quantization of this theory and the correspondence with the dual gauge theory leaving a more exhaustive analysis for the future [7].

[1] The relevant symmetry of the BMN sector is a super-Heisenberg algebra, which is a particular Inönü-Wigner contraction of the $SU(2,2|4)$ symmetry of $AdS_5\times S^5$. See for example [2].

[2] For instance, the action of non-relativistic bosonic string theory can be obtained from the method of non-linear realizations as a WZ term of the appropriate galilean group [5].

[3] The non-relativistic limit has been extended to the superbranes in flat space in [6], where the action for non-relativistic superstrings can be found. The symmetry of non-relativistic superstrings can be obtained as an Inönü-Wigner contraction of the $\mathcal{N}=2$ super-Poincare symmetry of flat space.

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The plan of the rest of the paper is as follows. In section two we introduce the basic ingredients of the Green-Schwarz action that are necessary to derive the worldsheet theory of non-relativistic string theory in $\mathrm{AdS}_5\times\ \mathrm{S}^5$. Section three explains the nature of the non-relativistic limit of string theories and the precise non-relativistic limit in $\mathrm{AdS}_5\times\ \mathrm{S}^5$ is presented. In section four we implement the non-relativistic limit directly in the worldsheet action and derive the formula for the worldsheet action describing non-relativistic strings. In section five we fix κ-symmetry and show that the worldsheet action becomes *free*! Section six shows that non-relativistic string theory in $\mathrm{AdS}_5\times\ \mathrm{S}^5$ is related by T-duality to a five-dimensional time dependent pp-wave. Section seven contains preliminary remarks about the description of non-relativistic strings in the dual $\mathcal{N}=4$ theory. We have relegated to the various appendices useful formulae that are used in the main text.

2. Superstring action in $\mathrm{AdS}_5 \times \mathrm{S}^5$

In this section we introduce the basic ingredients that are needed to derive the worldsheet theory describing non-relativistic strings in $\mathrm{AdS}_5\times\ \mathrm{S}^5$ as a limit of the sigma model on $\mathrm{AdS}_5\times\ \mathrm{S}^5$. The worldsheet lagrangian of type-IIB string theory in an arbitrary curved background is given by

$$\mathcal{L} = -\frac{T}{2}\left(\mathcal{L}^{(\mathrm{Kin})} + 2\mathcal{L}^{(WZ)}\right), \tag{2.1}$$

where T is the string tension, $\mathcal{L}^{(\mathrm{Kin})}$ is the kinetic term and $\mathcal{L}^{(WZ)}$ is the Wess-Zumino term. The corresponding action is invariant under worldsheet diffeomorphisms, space-time supersymmetry and, when the type-IIB supergravity constraints are satisfied, under κ-symmetry.

The worldsheet action in the $\mathrm{AdS}_5\times\ \mathrm{S}^5$ background can be formulated in a manifestly supersymmetric way — invariant under $\mathrm{SU}(2,2|4)$ — by writing it in terms of Cartan's one forms on the coset space

$$\frac{\mathrm{SU}(2,2|4)}{\mathrm{SO}(4,1)\times \mathrm{SO}(5)} \tag{2.2}$$

describing the $\mathrm{AdS}_5\times\ \mathrm{S}^5$ superspace. They are given by

$$\Omega = -ig^{-1}dg = P_m\mathbf{L}^m + P_{m'}\mathbf{L}^{m'} + Q_{\alpha\alpha' I}\mathbf{L}^{\alpha\alpha' I} + \frac{1}{2}M_{mn}\mathbf{L}^{mn} + \frac{1}{2}M_{m'n'}\mathbf{L}^{m'n'}\,, \tag{2.3}$$

where $P_m(P_{m'})$ denote the translation generators in $\mathrm{AdS}_5(\mathrm{S}^5)$, $M_{mn}(M_{m'n'})$ are the corresponding rotation generators and $Q_{\alpha\alpha' I}$ are the generators of supersymmetry. $m(\alpha)$ is a vector(spinor) index of the $\mathrm{SO}(4,1)$ tangent space symmetry of AdS_5, $m'(\alpha')$ a vector(spinor) index of the $\mathrm{SO}(5)$ tangent space symmetry of S^5 and I is a vector index of the $\mathrm{SL}(2,\mathbb{R})$ symmetry of type-IIB supergravity. In this language, the $\mathrm{SU}(2,2|4)$ algebra is encoded in Cartan's equation:[4]

$$d\Omega + i\Omega\wedge\Omega = 0\,. \tag{2.4}$$

[4]See appendix F for the commutation relations.

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In order to construct an action that is invariant under SU$(2,2|4)$, all that is required is to construct out of the Cartan one forms invariants under the stability[5] subgroup SO$(4,1)\times$ SO(5). One is left with [8]

$$\begin{aligned}\mathcal{L}^{(\mathrm{Kin})} &= \sqrt{-h}h^{ij}\left(\eta_{mn}\mathbf{L}_i^m\mathbf{L}_j^n+\delta_{m'n'}\mathbf{L}_i^{m'}\mathbf{L}_j^{n'}\right)\\ d(\mathcal{L}^{(WZ)}d^2\xi) &= i\,\bar{\mathbf{L}}(\Gamma_m\mathbf{L}^m+\Gamma_{m'}\mathbf{L}^{m'})\tau_3\mathbf{L}\,,\end{aligned}\tag{2.5}$$

where $\mathbf{L}_i^{m,m'}$ is the pullback of the one form on the worldsheet, h_{ij} the auxiliary two dimensional metric and ξ^i the worldsheet coordinates. The corresponding action is invariant under worldsheet diffeomorphisms, local Weyl transformations, SU$(2,2|4)$ and κ-symmetry.[6]

We also find it useful to consider the worldsheet action in the Nambu-Goto formulation. The Nambu-Goto action is obtained by integrating out the auxiliary metric h_{ij} in (2.5). In doing this, only the kinetic term is modified and in the normalization used in (2.1) reads

$$\mathcal{L}^{(\mathrm{Kin})} = 2\sqrt{-\det G_{ij}}\,,\tag{2.6}$$

where the induced metric is:

$$G_{ij}=\eta_{mn}\mathbf{L}_i^m\mathbf{L}_j^n+\delta_{m'n'}\mathbf{L}_i^{m'}\mathbf{L}_j^{n'}\,.\tag{2.7}$$

We can turn one more coupling on the worldsheet consistent with all the symmetries of the Green-Schwarz action. From the space-time point of view, it corresponds to turning on a closed B-field, which does not modify the supergravity equations of motion.

Therefore, the worldsheet lagrangian we are going to analyze is

$$\mathcal{L}=-\frac{T}{2}\left(\mathcal{L}^{(\mathrm{Kin})}+2\mathcal{L}^{(WZ)}+2\mathcal{L}^{B}\right)\tag{2.8}$$

with

$$\mathcal{L}^B=f^*B\,,\tag{2.9}$$

where f^*B is the pullback of B onto the worldsheet. This coupling is useful in constructing the worldsheet action of non-relativistic string theory in AdS$_5\times$ S^5.

In the Nambu-Goto formulation, worldsheet diffeomorphisms give rise to the following first class constraints

$$\begin{aligned}H &\equiv \frac{1}{2T}\left[\eta^{mn}\tilde{p}_m\tilde{p}_n+\delta^{m'n'}\tilde{p}_{m'}\tilde{p}_{n'}+T^2\left(\eta_{mn}\mathbf{L}_1^m\mathbf{L}_1^n+\delta_{m'n'}\mathbf{L}_1^{m'}\mathbf{L}_1^{n'}\right)\right]=0\,,\\ T &\equiv \tilde{p}_m\mathbf{L}_1^m+\tilde{p}_{m'}\mathbf{L}_1^{m'}=0\,,\end{aligned}\tag{2.10}$$

where $\tilde{p}_. \equiv \partial\mathcal{L}^{(\mathrm{Kin})}/\partial\dot{\mathbf{L}_0}$ corresponds to the mechanical momentum.[7] We note that the constraint $H\equiv 0$ is quadratic in the energy, which is characteristic of relativistic theories. As we show in this paper, the corresponding constraint for non-relativistic string

[5]This follows from the well known fact that Cartan's one forms on a coset space G/H are not invariant under the left action of G. The left action of G induces a transformation such that the one forms transform as gauge connections of H, that is $\Omega\to h\Omega h^{-1}-ihdh^{-1}$, where $h\in H$.

[6]The coefficient of $\mathcal{L}^{(WZ)}$ in (2.5) is determined by enforcing that the complete action (2.1) is invariant under κ-symmetry.

[7]The canonical momentum differs from the mechanical momentum $\tilde{p}$ by an additive amount proportional to the B field and by the contribution from the WZ term, generalizing the usual point particle relation $p=\tilde{p}+A$, where A is a background gauge field.

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theory (4.23) is linear in the energy, which is characteristic of non-relativistic theories. There are additional fermionic first class constraints associated with κ-symmetry, whose explicit expression we do not need in the paper.

In the Polyakov formulation, if we fix worldsheet reparametrizations by choosing the conformal gauge $\sqrt{-h}h_{ij} = \eta_{ij}$, the diffeomorphism constraints (2.10) imply the familiar Virasoro conditions

$$\begin{aligned} G_{00} + G_{11} &= 0 \\ G_{01} &= 0\,, \end{aligned} \tag{2.11}$$

where G_{ij} is the induced metric given in (2.7).

Thus far we have written the action without a specific choice of supercoordinates on the coset, or equivalently we did not invoke a specific choice of coset representative g. In the next section, we choose a parametrization which is conducive to taking the non-relativistic limit.

3. Non-relativistic limit of $\mathrm{AdS}_5 \times \mathrm{S}^5$

In this section we find a limit of string theory describing non-relativistic strings in $\mathrm{AdS}_5\times\mathrm{S}^5$. Non-relativistic string theory[8] can be defined by taking a suitable non-relativistic limit of relativistic string theory. The basic idea is to find a limit where the "light" strings satisfy a non-relativistic dispersion relation while the rest of the degrees of freedom become infinitely "heavy". This can be accomplished by considering wound (charged) strings in the presence of a background B-field and tuning the B-field so that the energy coming from the background B-field cancels the tension of the string.[9] In this way, only closed strings with positive charge remain "light", while strings with non-positive charge become infinitely "heavy" in the limit. Therefore, the limit eliminates the anti-particles, whose absence is characteristic of non-relativistic theories. The winding number plays the role of the mass of the non-relativistic string.

The remaining "light" strings obey a non-relativistic dispersion relation which can be derived by either solving the Virasoro constraints [3] or by solving the diffeomorphism constraints [13], which are linear in the energy and thus non-relativistic in nature. Furthermore, in this limit the speed of light in the directions transverse to the string is sent to infinity, giving rise to instantaneous action at a distance, which is also a characteristic property of non-relativistic point particle theories.

In order to take the limit, we must specify the coordinates we are going to use to parametrize the target space. The coordinates of $\mathrm{AdS}_5\times\mathrm{S}^5$ are determined by a choice of

[8]In this section we briefly summarize some basic facts about non-relativistic strings (see [3] for more details and explanations).

[9]The same limit can be taken for open strings ending on a D-brane, giving rise to the NCOS theory [9, 10]. It was in this context where the appearance of non-relativistic strings first surfaced [11]. One can also consider open strings on a D-brane transverse to the B-field, which gives rise to non-relativistic open strings [12].

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parametrization of the SU(2, 2|4)/(SO(4, 1) × SO(5)) coset space. We consider the following one

$$g = e^{iP_1 X^1} e^{iP_0 X^0} e^{iP_a X^a} e^{iP_{m'} X^{m'}} e^{iQ_{\alpha\alpha' I}\theta^{\alpha\alpha' I}} , \tag{3.1}$$

where the index m splits into (μ, a) with $\mu = 0, 1$ and $a = 2, 3, 4$. Therefore, the coordinates of $AdS_5 \times S^5$ are given by (X^μ, X^a) and $X^{m'}$ respectively, while $\theta^{\alpha\alpha' I}$ are the corresponding fermionic coordinates. This parametrization makes manifest an embedding of AdS_2 in AdS_5, where X^μ are the coordinates of AdS_2. A specific choice of an AdS_2 representative is selected to make manifest in the associated AdS_2 metric a space-like isometry along which the non-relativistic strings can wind.

Using the parametrization (3.1) and the SU(2, 2|4) algebra, we can compute the invariant one forms (2.3). The metric of $AdS_5 \times S^5$ is obtained from the bosonic components of the one-forms associated with translations, which correspond geometrically to the metric vielbeins on the coset:[10]

$$\begin{aligned} \mathbf{L}^m &= \mathbf{e}^m + \text{fermions} \implies ds^2_{AdS} = \mathbf{e}^m \mathbf{e}^m \\ \mathbf{L}^{m'} &= \mathbf{e}^{m'} + \text{fermions} \implies ds^2_{S} = \mathbf{e}^{m'} \mathbf{e}^{m'} . \end{aligned} \tag{3.2}$$

Using the formulae in appendix A, the metric of AdS_5 in these coordinates is given by

$$\begin{aligned} ds^2_{AdS} = {} & \cosh^2\rho \left[-(dX^0)^2 + \cos^2\left(\frac{X^0}{R}\right)(dX^1)^2 \right] + \\ & + \left[\left(\frac{\sinh\rho}{\rho}\right)^2 (dX^a)^2 - \left(\frac{\sinh^2\rho}{\rho^2} - 1\right) \frac{(X_a dX^a)^2}{\rho^2 R^2} \right] , \end{aligned} \tag{3.3}$$

where $\rho = \sqrt{X_a X^a}/R$ and R is the radius of AdS_5 (S^5), while the metric inside the first bracket is that of AdS_2 (See appendix B for details of the corresponding geometry). We note that in these coordinates, $\frac{\partial}{\partial X^1}$ is a(n almost everywhere) space-like Killing vector, so that a string can wrap along this coordinate. By appropriately tuning the background B-field the wound closed strings can be made "light" and non-relativistic.

Likewise, the S^5 metric in the coordinates we are using is given by

$$ds^2_S = \left(\frac{\sin r}{r}\right)^2 (dX^{m'})^2 - \left(\frac{\sin^2 r}{r^2} - 1\right) \frac{(X_{m'} dX^{m'})^2}{r^2 R^2} , \tag{3.4}$$

where $r = \sqrt{X_{m'} X^{m'}}/R$.

The non-relativistic limit rescales the longitudinal coordinates X^μ differently than the rest. Physically, this corresponds to analyzing a string that spans an AdS_2 subspace inside $AdS_5 \times S^5$, such as a string winding along the X^1 coordinate. In order to get a supersymmetric and κ-symmetric action we must also appropriately rescale the fermionic coordinates in a way consistent with the symmetries preserved by the bosonic scaling. These conditions single out a unique scaling for $\theta^{\alpha\alpha' I}$.

[10]We recall that the bosonic component of the one-forms $\mathbf{L}^{mn}, \mathbf{L}^{m'n'}$ associated with the generators of the stabilizer group of the coset, correspond geometrically to the spin connections $w^{mn}(w^{m'n'})$ of $AdS_5(S^5)$, where $\mathbf{L}^{mn} = w^{mn} + \text{fermions}$ and $\mathbf{L}^{m'n'} = w^{m'n'} + \text{fermions}$.

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In order to obtain a well defined Green-Schwarz action after taking the non-relativistic limit, we introduce a closed B-field along AdS_2. As explained above, the limit involves tuning the B-field so that it cancels the divergent contribution to the energy coming from the area of the worldsheet and the WZ term, so that one is left with non-relativistic strings of finite energy. The explicit worldsheet coupling (2.9) is given by

$$\mathcal{L}^B = B_{\mu\nu}\mathbf{v}_0^\mu \mathbf{v}_1^\nu \,, \tag{3.5}$$

where $\mathbf{v}^\mu$ are the zweibeins[11] of the AdS_2 subspace spanned by X^μ and $B_{\mu\nu}$ is a constant anti-symmetric matrix.

Having motivated the physics, we can now write down the non-relativistic limit of string theory in $AdS_5\times S^5$. It is given by:

$$\begin{aligned} X^\mu &= \omega x^\mu \,, \qquad \theta = \sqrt{\omega}\,\theta_- + \frac{1}{\sqrt{\omega}}\,\theta_+ \\ B_{\mu\nu} &= \epsilon_{\mu\nu} \,, \qquad R = \omega R_0 \end{aligned} \tag{3.6}$$

and then take $\omega \to \infty$.

In the non-relativistic limit, the longitudinal coordinates of the string are rescaled while the transverse ones are left untouched, which implies that the transverse fluctuations are small. This also has the effect of making the speed of light in the transverse directions[12] very large, infinite in the limit. The radius R is also sent to infinity, which has the effect of flattening out the transverse space to AdS_2. The B-field is tuned in such a way that its contribution to the energy is precisely cancelled by the contribution coming from the tension of the string. We shall see later that there is a very precise sense in which the limit is non-relativistic. As we shall see, the worldsheet theory that we obtain by taking the limit (3.6) is invariant under the supersymmetric version of the Newton-Hooke group [14], which captures the non-relativistic kinematics of slowly moving particles in AdS.

The scaling behaviour of the fermionic coordinates in (3.6) is characterized by their transformation properties under the matrix Γ_*, which splits the fermionic coordinates into two eigenspaces with eigenvalues ± 1

$$\Gamma_*\theta_\pm = \pm\theta_\pm \qquad \text{where } \Gamma_* = \Gamma_0\Gamma_1\tau_3 \tag{3.7}$$

and $\Gamma_*^2 = 1$. As shown in appendix E, the matrix Γ_* is the first term in the non-relativistic expansion of the matrix Γ_κ appearing in the κ-symmetry transformations of the relativistic string action.

Having physically motivated the limit, we are now ready to analyze its consequences.

4. Non-relativistic string theory

In this section we derive the worldsheet action of non-relativistic strings in $AdS_5\times S^5$ by taking the non-relativistic limit of the Green-Schwarz action in $AdS_5\times S^5$ described in the

[11] The AdS_2 zweibein one forms $\mathbf{v}^\mu$ can be related to the AdS_5 vielbeins $\mathbf{e}^\mu$ by $\mathbf{e}^\mu = \cosh\rho\, \mathbf{v}^\mu$.

[12] The speed of light along the string is unchanged.

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previous section. The rigid and local symmetries of the resulting action are also discussed and the role played by the supersymmetric extension of the Newton-Hooke group — the kinematical group of non-relativistic AdS — is explained.

The basic idea is to take the non-relativistic limit (3.6) directly in the worldsheet action and obtain in this way a definition of the non-relativistic theory. Throughout our analysis, we keep ω large but finite in the intermediate computations and only send ω strictly to infinity at the end. Therefore, we keep explicitly terms in the action that scale as positive powers of ω (which look superficially divergent) and terms that are independent of ω (which are finite). We drop terms that scale as inverse powers of ω since they have no chance of contributing when taking the limit at the end of the analysis.

In order to analyze the behaviour of the sigma model in $AdS_5\times S^5$ in the non-relativistic limit, we must first compute how the various Cartan one-forms behave under the scaling (3.6). Using the expressions in appendix C, it is straightforward to show that the one forms relevant for writing the string action (2.8) scale as follows

$$\begin{aligned} \mathbf{L}^{\mu} &= \omega \mathbf{L}^{\mu(1)} + \frac{1}{\omega}\mathbf{L}^{\mu(-1)} + \cdots \\ \mathbf{L}^{a} &= \mathbf{L}^{a(0)} + \cdots \\ \mathbf{L}^{m'} &= \mathbf{L}^{m'(0)} + \cdots \\ \mathbf{L}^{\alpha\alpha' I} &= \sqrt{\omega}\,\mathbf{L}^{\alpha\alpha' I(1/2)} + \frac{1}{\sqrt{\omega}}\mathbf{L}^{\alpha\alpha' I(-1/2)} + \cdots , \end{aligned} \tag{4.1}$$

where $\mathbf{L}^{\cdot(n)}$ is the term scaling as ω^n in the form $\mathbf{L}^{\cdot}$ and $\cdots$ refer to terms in the one form which do not contribute to the action in the $\omega \to \infty$ limit. Given the expansion of the Cartan one forms in powers in ω we can look at the worldsheet action (2.8) and identify the terms which are finite and the terms which are superficially divergent.

The finite contribution coming from the Polyakov kinetic term[13] (2.5) in (2.8) is given by

$$\mathcal{L}^{(\mathrm{Kin})}_{\mathrm{fin}} = \sqrt{-h}h^{ij}\left(2\eta_{\mu\nu}\mathbf{L}_i^{\mu(1)}\mathbf{L}_j^{\nu(-1)} + \delta_{ab}\mathbf{L}_i^{a(0)}\mathbf{L}_j^{b(0)} + \delta_{m'n'}\mathbf{L}_i^{m'(0)}\mathbf{L}_j^{n'(0)}\right), \tag{4.2}$$

while the superficially divergent contribution is given by:

$$\mathcal{L}^{(\mathrm{Kin})}_{\mathrm{div}} = \omega^2\sqrt{-h}h^{ij}\eta_{\mu\nu}\mathbf{L}_i^{\mu(1)}\mathbf{L}_j^{\nu(1)}\,. \tag{4.3}$$

The finite contribution from the WZ term (2.5) in (2.8) can be written as

$$\begin{aligned} \frac{1}{i}d(\mathcal{L}^{(WZ)}_{\mathrm{fin}}d^2\xi) = {}& \bar{\mathbf{L}}^{(1/2)}\Gamma_{\mu}\mathbf{L}^{\mu(-1)}\tau_3\mathbf{L}^{(1/2)} + \bar{\mathbf{L}}^{(-1/2)}\Gamma_{\mu}\mathbf{L}^{\mu(1)}\tau_3\mathbf{L}^{(-1/2)} + \\ & + 2\bar{\mathbf{L}}^{(1/2)}\Gamma_a\mathbf{L}^{a(0)}\tau_3\mathbf{L}^{(-1/2)} + 2\bar{\mathbf{L}}^{(1/2)}\Gamma_{m'}\mathbf{L}^{m'(0)}\tau_3\mathbf{L}^{(-1/2)}\,, \end{aligned} \tag{4.4}$$

while the superficially divergent contribution is:

$$\frac{1}{i}d(\mathcal{L}^{(WZ)}_{\mathrm{div}}d^2\xi) = \omega^2\,\bar{\mathbf{L}}^{(1/2)}\Gamma_{\mu}\mathbf{L}^{\mu(1)}\tau_3\mathbf{L}^{(1/2)}. \tag{4.5}$$

[13] An analogous analysis can be performed in the Nambu-Goto formulation. See appendix E for a detailed analysis of κ-symmetry.

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The final term to consider is the coupling (3.5) corresponding to turning on a closed B-field, which only leads to the following potentially divergent term

$$\mathcal{L}^{B}_{\text{div}} = \omega^2 \epsilon_{\mu\nu} \mathbf{e}_0^{\mu(1)} \mathbf{e}_1^{\nu(1)}\,, \tag{4.6}$$

where we have used that $\mathbf{v}^\mu = \mathbf{e}^{\mu(1)}$, where $\mathbf{L}^{\mu(1)} = \mathbf{e}^{\mu(1)} + \mathbf{L}^{\mu(1)}_{\text{fermionic}}$.

In order to give a proper definition of non-relativistic string theory in $\text{AdS}_5\times\text{S}^5$ we must rewrite the superficially divergent terms in the action

$$\mathcal{L}_{\text{div}} = -\frac{T}{2}\left(\omega^2\sqrt{-h}h^{ij}\eta_{\mu\nu}\mathbf{L}_i^{\mu(1)}\mathbf{L}_j^{\nu(1)} + 2\mathcal{L}^{(WZ)}_{\text{div}} + 2\omega^2\epsilon_{\mu\nu}\mathbf{e}_0^{\mu(1)}\mathbf{e}_1^{\nu(1)}\right) \tag{4.7}$$

in a way that the $\omega \to \infty$ limit yields a consistent worldsheet theory, which gives a proper definition of the theory. We will now show that there is a rich interplay between the various terms in (4.7) such that when combined, conspire to yield a well defined worldsheet action, which serves as the definition of non-relativistic string theory in $\text{AdS}_5\times$ S^5.

In order to show this we first note that the following identity can be proven[14]

$$d(\omega^2(\det \mathbf{L}^{\mu(1)})_{\text{fermionic}} + \mathcal{L}^{(WZ)}_{\text{div}}) = 0\,, \tag{4.8}$$

where $(\det \mathbf{L}^{\mu(1)})_{\text{fermionic}}$ is defined as follows:

$$\det \mathbf{L}^{\mu(1)} = \det \mathbf{e}^{\mu(1)} + (\det \mathbf{L}^{\mu(1)})_{\text{fermionic}}\,. \tag{4.9}$$

This identity, which is proven in appendix D, guarantees that the superficially divergent term of the Nambu-Goto lagrangian is a total derivative. This divergence is precisely cancelled by the background B-field.

Formula (4.8) implies that up to an exact form one has:

$$\mathcal{L}^{(WZ)}_{\text{div}} = -\omega^2(\det \mathbf{L}^{\mu(1)})_{\text{fermionic}}\,. \tag{4.10}$$

Therefore, if we combine the divergent piece of the WZ term with the divergent piece coming from the B-field, the Polyakov action can be written in terms of the Cartan one forms $\mathbf{L}^{\mu(1)}$

$$\mathcal{L}^{(WZ)}_{\text{div}} + \mathcal{L}^{B}_{\text{div}} = -\omega^2 \det \mathbf{L}^{\mu(1)}\,, \tag{4.11}$$

where we have used that $\det \mathbf{e}^{\mu(1)} = -\epsilon_{\mu\nu}\mathbf{e}_0^{\mu(1)}\mathbf{e}_1^{\nu(1)}$ and formula (4.9).

After these manipulations, we have that the superficially divergent part of the Polyakov action (4.7) can be written as:

$$\mathcal{L}_{\text{div}} = -\omega^2\,\frac{T}{2}\left(\sqrt{-h}h^{ij}\eta_{\mu\nu}\mathbf{L}_i^{\mu(1)}\mathbf{L}_j^{\nu(1)} - 2\det \mathbf{L}^{\mu(1)}\right). \tag{4.12}$$

It is straightforward to show that these two terms combine to complete a perfect square

$$\mathcal{L}_{\text{div}} = -\omega^2\,\frac{T}{2}\sqrt{-h}h^{00}\eta_{\mu\nu}\,f^\mu\,f^\nu\,, \tag{4.13}$$

[14] Our convention is $\epsilon_{01} = -\epsilon^{01} = -1$.

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where:

$$f^\mu \equiv \left[\mathbf{L}_0^{\mu(1)} - \frac{\sqrt{-h}}{h_{11}}\epsilon^{\mu\rho}\eta_{\rho\sigma}\mathbf{L}_1^{\sigma(1)} - \frac{h_{01}}{h_{11}}\mathbf{L}_1^{\mu(1)}\right]. \tag{4.14}$$

We can now rewrite this superficially divergent term in a way that the $\omega \to \infty$ limit can be taken smoothly. The idea is to introduce Lagrange multipliers λ_μ to rewrite the action as follows

$$\mathcal{L}_{\rm div} = \lambda_\mu f^\mu + \frac{1}{2\sqrt{-h}h^{00}T\omega^2}\lambda_\mu\lambda^\mu\,, \tag{4.15}$$

which reproduces (4.13) by integrating out the λ_μ variables. Once the extra variables λ_μ are introduced, we can take the strict non-relativistic limit $\omega \to \infty$ in (4.15) and be left with a finite contribution:

$$\mathcal{L}^* = \lambda_\mu f^\mu\,. \tag{4.16}$$

Now that we have properly defined the superficial divergence, we can finally write the complete action for non-relativistic strings in $\mathrm{AdS}_5\times\mathrm{S}^5$

$$\begin{aligned}\mathcal{L} &= -\frac{T}{2}\left(\mathcal{L}_{\rm fin}^{(\rm Kin)} + 2\mathcal{L}_{\rm fin}^{(WZ)}\right) + \mathcal{L}^* \\ &= -\frac{T}{2}\sqrt{-h}h^{ij}\mathbf{G}_{ij}^{nr} - T\mathcal{L}_{\rm fin}^{(WZ)} + \lambda_\mu\left[\mathbf{L}_0^{\mu(1)} - \frac{\sqrt{-h}}{h_{11}}\epsilon^{\mu\rho}\eta_{\rho\sigma}\mathbf{L}_1^{\sigma(1)} - \frac{h_{01}}{h_{11}}\mathbf{L}_1^{\mu(1)}\right],\end{aligned} \tag{4.17}$$

where

$$\mathbf{G}_{ij}^{nr} = \eta_{\mu\nu}(\mathbf{L}_i^{\mu(1)}\mathbf{L}_j^{\nu(-1)} + \mathbf{L}_j^{\mu(1)}\mathbf{L}_i^{\nu(-1)}) + \delta_{ab}\mathbf{L}_i^{a(0)}\mathbf{L}_j^{b(0)} + \delta_{m'n'}\mathbf{L}_i^{m'(0)}\mathbf{L}_j^{n'(0)} \tag{4.18}$$

and the expression for $\mathcal{L}_{\rm fin}^{(WZ)}$ is determined by integration of the three form (4.4). We can now ascribe the variables λ_μ with a physical interpretation; they are linearly related to the momentum along the longitudinal directions, as can be seen from (4.17).

The gauge symmetries of the action[15] are generated by worldsheet diffeomorphisms, Weyl transformations and κ-transformations. The existence of these symmetries can be understood as being a consequence of the existence of the corresponding symmetries of the parent relativistic theory. The limit we have found maps a symmetry of the parent theory to a corresponding symmetry of the non-relativistic theory.[16]

The action (4.17) is also invariant under a supersymmetric version of the Newton-Hooke group with thirty two supersymmetries! The Newton-Hooke group is precisely the AdS analog of the galilean group for flat space. It can be obtained by an Inönü-Wigner contraction of the $SO(2,4)$ AdS symmetry in the limit in which the speed of light is sent to infinity, which is precisely how the galilean group can be obtained from the Poincare symmetry group of flat space. The only difference between the Newton-Hooke group and the galilean group is that in the latter the hamiltonian commutes with translations, while in the former one has $[H, P_a] = i/R_0^2\,K_a$ [14], where K_a generate boosts between inertial

[15]The structure of the Polyakov form of the lagrangian (4.17) for the bosonic string in flat space has been discussed in [15], where it was derived from the Nambu-Goto action.

[16]Appendix E shows that the symmetries of the parent theory have a counterpart in the theory after the limit.

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frames. The commutation relations satisfied by the generators of the supersymmetric Newton-Hooke algebra can be found in appendix F, which can be obtained by an Inönü-Wigner contraction of the $SU(2,2|4)$ algebra.

The non-relativistic Polyakov action (4.17) is invariant under diffeomorphisms, albeit it is not manifest from the form of the action. We can also write down the action of non-relativistic string theory in $AdS_5\times S^5$ in the Nambu-Goto formulation, where invariance under diffeomorphisms is manifest. This can be done by using the equation of motion of λ_μ. Using it we get:

$$h_{ij} \propto \mathbf{L}_i{}^{\mu(1)}\mathbf{L}_j{}^{\nu(1)}\eta_{\mu\nu} \equiv \mathbf{g}_{ij}\,. \tag{4.19}$$

Plugging this into (4.17) one derives the Nambu-Goto lagrangian of non-relativistic string theory in $AdS_5\times S^5$

$$\mathcal{L} = -\frac{T}{2}\sqrt{-\mathbf{g}}\mathbf{g}^{ij}\mathbf{G}^{nr}_{ij} - T\mathcal{L}^{(WZ)}_{\text{fin}}\,, \tag{4.20}$$

where $\mathbf{G}^{nr}_{ij}$ is the induced metric on the worldsheet (4.18). This is the lagrangian of non-relativistic strings in $AdS_5\times S^5$ in the Nambu-Goto formulation.

From the action we can derive the constraints associated with worldsheet diffeomorphisms. Using the mechanical momenta

$$\hat{p}^+_\mu = \frac{\partial\mathcal{L}^{(NG)}}{\partial\mathbf{L}_0^{\mu(1)}}\,,\qquad \hat{p}_a = \frac{\partial\mathcal{L}^{(NG)}}{\partial\mathbf{L}_0^{a(0)}}\,,\qquad \hat{p}_{m'} = \frac{\partial\mathcal{L}^{(NG)}}{\partial\mathbf{L}_0^{m'(0)}} \tag{4.21}$$

and

$$\hat{p}^-_\mu = \frac{\partial\mathcal{L}^{(NG)}}{\partial\mathbf{L}_0^{\mu(-1)}} = T\epsilon_{\mu\nu}\mathbf{L}_1^{\mu(1)} \tag{4.22}$$

we can derive the following identities:

$$\begin{aligned} H_{nr} \equiv{}& \hat{p}^+_\mu\varepsilon^{\mu\rho}\eta_{\rho\sigma}\mathbf{L}_1^{\sigma(1)} + \hat{p}^-_\mu\varepsilon^{\mu\rho}\eta_{\rho\sigma}\mathbf{L}_1^{\sigma(-1)} + 2T\eta_{\mu\nu}\mathbf{L}_1{}^{\mu(1)}\mathbf{L}_1{}^{\nu(-1)} + \\ &+\frac{1}{2T}\left[\delta^{ab}\hat{p}_a\hat{p}_b + \delta^{m'n'}\hat{p}_{m'}\hat{p}_{n'} + T^2\left(\delta_{ab}\mathbf{L}_1^{a(0)}\mathbf{L}_1^{b(0)} + \delta_{m'n'}\mathbf{L}_1^{m'(0)}\mathbf{L}_1^{n'(0)}\right)\right] = 0\,, \end{aligned} \tag{4.23}$$

$$T_{nr} \equiv \hat{p}^+_\mu\mathbf{L}_1{}^{\mu(1)} + \hat{p}^-_\mu\mathbf{L}_1{}^{\mu(-1)} + \hat{p}_a\mathbf{L}_1{}^{a(0)} + \hat{p}_{m'}\mathbf{L}_1{}^{m'(0)} = 0\,. \tag{4.24}$$

They are diffeomorphism constraints when written in terms of the canonical momenta. They can also be obtained from the corresponding relativistic ones in (2.10) by taking the non-relativistic limit. We note that the constraint H_{nr} is now *linear* in the energy, which is what one expects from a non-relativistic theory.

5. Gauge fixed action

In this section we find that the non-relativistic string action becomes free by suitably fixing all the gauge symmetries. For explicit computations, it is convenient to choose the conformal gauge $\sqrt{-h}h^{ij} = \eta^{ij}$. Then the Polyakov action (4.17) simplifies to:

$$\mathcal{L} = -\frac{T}{2}\eta^{ij}\mathbf{G}^{nr}_{ij} - T\mathcal{L}^{(WZ)}_{\text{fin}} + \lambda_\mu(\mathbf{L}_0^{\mu(1)} - \epsilon^{\mu\rho}\eta_{\rho\sigma}\mathbf{L}_1^{\sigma(1)})\,. \tag{5.1}$$

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In the gauge fixed form, the term in (5.1) involving λ_μ is the $\mathrm{AdS}_5\times\mathrm{S}^5$ analog of the (β,γ) system introduced in [3] to describe non-relativistic string theory in flat space. In fact, in the $R_0\to\infty$ limit, we recover the result for the non-relativistic superstring in the flat space [6].

In order to completely define the theory in the conformal gauge, we must supplement the action with the Virasoro constraints, which are given by

$$\begin{aligned}\lambda_\mu\epsilon^{\mu\nu}\eta_{\nu\rho}\mathbf{L}_1^{\rho(1)}+\frac{T}{2}(\mathbf{G}^{nr}_{00}+\mathbf{G}^{nr}_{11})&=0\\ \lambda_\mu\mathbf{L}_1^{\mu(1)}+T\mathbf{G}^{nr}_{01}&=0\,,\end{aligned}\tag{5.2}$$

where $\mathbf{G}^{nr}_{ij}$ is defined in (4.18).

A simplification of the action can be achieved by fixing a gauge for κ-symmetry. In appendix E we show that we can fix κ-symmetry by making the following gauge choice:

$$\theta_-=0\,.\tag{5.3}$$

By fixing this gauge, the lagrangian simplifies dramatically since many terms automatically vanish, as can be seen by looking at the expressions for the one forms in appendix C. Moreover, we can integrate the three-form (4.4) and obtain an explicit expression for the WZ term.

By using the expressions in appendix 4 we have that after fixing κ-symmetry, the Polyakov action (5.1) can be written as:

$$\begin{aligned}\mathcal{L}^{(NR)}&=-T\Big[\frac{\eta^{ij}}{2}\partial_i x^a\partial_j x^b\eta_{ab}+\frac{\mathbf{g}_{11}-\mathbf{g}_{00}}{2\,R_0^2}x^a x^b\delta_{ab}+\frac{\eta^{ij}}{2}\partial_i x^{m'}\partial_j x^{n'}\delta_{m'n'}-\\ &\qquad-i\bar\theta_+\Gamma^\mu(\eta_{\mu\nu}\eta^{ij}\mathbf{v}_j{}^\nu)\mathbf{D}_i\theta_+-i\bar\theta_+\Gamma^\mu(\det\mathbf{v}_j{}^\nu)\mathbf{v}_\mu{}^i\mathbf{D}_i\theta_+\Big]+\\ &\quad+\lambda_\mu(\mathbf{v}_0^\mu-\epsilon^{\mu\rho}\eta_{\rho\nu}\mathbf{v}_1^\nu)\,.\end{aligned}\tag{5.4}$$

$\mathbf{g}_{ij}=\eta_{\mu\nu}\mathbf{v}_i{}^\mu\mathbf{v}_j{}^\nu$ is the metric of AdS_2 spanned by the longitudinal coordinates written in terms of the worldsheet coordinates ξ^i. The lagrangian has two fermionic pieces arising respectively from the Kinetic and WZ term. The covariant derivative $\mathbf{D}_i\theta_+$ is given by

$$\mathbf{D}\theta_+=\left(d+\frac{1}{2R_0}\mathbf{v}^\mu\Gamma_\mu\sigma_1\tau_2+\frac{1}{4}\mathbf{w}^{\mu\nu}\Gamma_{\mu\nu}\right)\theta_+\,,\tag{5.5}$$

where in the coordinates we are using

$$\mathbf{v}^\mu=\left(dx^0,dx^1\cos\frac{x^0}{R_0}\right),\qquad \mathbf{w}^{01}=-\frac{dx^1}{R_0}\sin\frac{x^0}{R_0}\,.\tag{5.6}$$

In the action (5.4), $\mathbf{v}_\mu{}^i$ is the inverse of $\mathbf{v}_i{}^\mu$.

The equation of motion for the Lagrange multipliers λ_μ is given by

$$\mathbf{v}_0^\mu-\epsilon^{\mu\rho}\eta_{\rho\nu}\mathbf{v}_1^\nu=0\,,\tag{5.7}$$

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which implies that the induced metric $\mathbf{g}_{ij}$ along the longitudinal coordinates is conformally flat:

$$\begin{aligned} \mathbf{g}_{00} + \mathbf{g}_{11} &= 0\,, \\ \mathbf{g}_{01} &= 0\,. \end{aligned} \tag{5.8}$$

Since we are using the non-relativistic string lagrangian in the gauge fixed form, we must still impose the corresponding Virasoro conditions (5.2). This guarantees equivalence with the diffeomorphism invariant theory.

We can also work with the reparametrization invariant Nambu-Goto lagrangian (4.20) where κ-symmetry is gauge fixed using $\theta_- = 0$. From (4.20) one obtains:

$$\begin{aligned} \mathcal{L}^{(NR)} = -T\sqrt{-\det \mathbf{g}}\Big[&\frac{\mathbf{g}^{ij}}{2}\partial_i x^a \partial_j x^b \eta_{ab} + \frac{1}{R_0^2} x^a x^b \delta_{ab} + \frac{\mathbf{g}^{ij}}{2}\partial_i x^{m'} \partial_j x^{n'} \delta_{m'n'} \\ &- 2i\,\overline{\theta}_+ \Gamma^\mu \mathbf{v}_\mu{}^i \mathbf{D}_i \theta_+\Big]\,. \end{aligned} \tag{5.9}$$

We note that in the $R_0 \to \infty$ limit, we recover the result for the non-relativistic superstring in flat space [6].

The lagrangian (5.9) is interacting since the longitudinal scalars $x^\mu(\xi)$ are coupled to the transverse scalars $x^a(\xi)$, $x^{m'}(\xi)$ and the dynamical fermions $\theta_+(\xi)$ via the induced AdS_2metric $\mathbf{g}_{ij}$. A further simplification occurs if we fix worldsheet diffeomorphisms by choosing the static gauge:

$$x^\mu(\xi) = \xi^\mu\,. \tag{5.10}$$

In this gauge (5.9) becomes a free field lagrangian. The theory describes a collection of scalars and fermions propagating in AdS_2. More precisely, we have five bosonic massless fields $x^{m'}$, three massive bosonic fields x^a with $m^2 = 2/R_0^2$ and sixteen massive fermions θ_+ with $m^2 = 1/R_0^2$. This same two dimensional field theory in AdS_2 has been studied in the past in [16 – 19].

Just as in relativistic string theory, once κ-symmetry is fixed, sixteen of the supersymmetries are linearly realized while the other sixteen are non-linearly realized. The non-linearly realized supersymmetries are generated by ϵ_+, which induce the following transformations:

$$\delta\theta_+ = K\epsilon_+\,, \qquad \delta x^\mu = \delta x^a = \delta x^{m'} = 0\,, \tag{5.11}$$

where

$$K = e^{-\frac{\Gamma_0 \sigma_1 \tau_2 x^0}{2R_0}} e^{-\frac{\Gamma_1 \sigma_1 \tau_2 x^1}{2R_0}}\,, \tag{5.12}$$

and satisfies

$$\mathbf{D}\,K = 0\,. \tag{5.13}$$

The linearly realized supersymmetries are induced by ϵ_-,[17] which generate the following transformations:

$$\delta x^a = -2i\overline{\theta}_+ \Gamma^a K\epsilon_-\,, \qquad \delta x^{m'} = -2i\overline{\theta}_+ \Gamma^{m'} K\epsilon_-\,, \qquad \delta x^\mu = 0 \tag{5.14}$$

[17] The relation between the relativistic susy parameters ϵ and the non-relativistic ones is given by $\epsilon = \sqrt{\omega}\epsilon_- + \frac{1}{\sqrt{\omega}}\epsilon_+$.

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$$\delta\theta_+ = \frac{1}{2}(\Gamma^\mu e_\mu{}^j)(\Gamma_a\partial_j x^a + \Gamma_{m'}\partial_j x^{m'})K\epsilon_- - \frac{1}{2R_0}(\Gamma_a x^a - \Gamma_{m'}x^{m'})\sigma_1\tau_2 K\epsilon_- \,. \quad (5.15)$$

Geometrically, $K\epsilon_\pm$ are the Killing spinors of the AdS_2 field theory in the basis of vielbeins given in (5.6).

To summarize, we have shown that the worldsheet action of non-relativistic string theory in $AdS_5\times S^5$ reduces to a free field theory in AdS_2 with maximal supersymmetry. Given the explicit form of the local action it is a very interesting problem to study the spectrum and interactions of non-relativistic strings [7].

6. T-dual description of non-relativistic strings

In section 4 we have found that string propagation in the non-relativistic limit (4.17) is regular even though the $AdS_5\times S^5$ background becomes singular in the limit. In particular, we found a way of rewriting the worldsheet action such that the result was finite. This can be accomplished because of a precise cancellation between the divergent contributions from the Kinetic and WZ terms with the B-field contribution. The crucial ingredient in accomplishing this was that the divergence originating from the singular background

$$ds^2 = (\omega^2 + \hat\rho^2)\left[-(dx^0)^2 + \cos^2\left(\frac{x^0}{R_0}\right)(dx^1)^2\right] + dx^a dx^a + dx^{m'}dx^{m'}, \qquad (6.1)$$

is precisely compensated by a divergence in the background B field

$$B = \omega^2 \cos\left(\frac{x^0}{R_0}\right) dx^0 \wedge dx^1 \,, \qquad (6.2)$$

where $\hat\rho = \sqrt{x_a x^a}/R_0$.

Just like for the non-relativistic limit in flat space, it is quite useful to analyze the fate of the background (6.1),(6.2) under T-duality. We have chosen the parametrization of the coset in (3.1) in such a way that the resulting metric (3.3) exhibits a manifest space-like isometry generated by the Killing vector $V = \frac{\partial}{\partial x^1}$. Let's compactify the coordinate x^1 on a circle of radius L. Once we have a finite space-like circle, we can perform a T-duality. The T-dual background can be computed by using the usual T-duality rules [20, 21]

$$\begin{aligned} G_{11} &= \frac{1}{g_{11}} \to 0 \\ G_{00} &= g_{00} - \frac{g_{01}^2 - B_{01}^2}{g_{11}} \to -2\hat\rho^2 \\ G_{01} &= \frac{B_{01}}{g_{11}} \to \frac{1}{\cos\left(\frac{x^0}{R_0}\right)} \end{aligned} \qquad (6.3)$$

while the rest of the metric components are left invariant. We note that the coordinate x^1 has now become a null coordinate which is compactified on a circle of radius α'/L. Note that after T-duality the background is regular!

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By making the change of coordinates

$$\sin\left(\frac{x^0}{R_0}\right) = \tanh\left(\frac{\widetilde{x}^0}{R_0}\right), \tag{6.4}$$

we find that the T-dual of the non-relativistic $\mathrm{AdS}_5\times\mathrm{S}^5$ background is the product of a five-dimensional pp-wave times R^5 where the pp-wave has a compact null coordinate. The metric of the T-dual background is:

$$ds^2 = 2d\widetilde{x}^0 dx^1 - \frac{2x^a x^a}{R_0^2\cosh^2\left(\frac{\widetilde{x}^0}{R_0}\right)} d\widetilde{x}^0 d\widetilde{x}^0 + dx^a dx^a + dx^{m'}dx^{m'}\,. \tag{6.5}$$

The background also contains a null RR flux and a dilaton giving rise to a solution of the type-IIA supergravity equations of motion. The geometrical relation between time dependent pp-waves and the Newton-Hooke was explored in [22].

We have shown that the T-dual description of non-relativistic string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ is given by a time dependent pp-wave with the null coordinate compactified. Therefore, quantization of non-relativistic strings in $\mathrm{AdS}_5\times\mathrm{S}^5$ is T-dual to the discrete light-cone quantization (DLCQ) of the pp-wave. The result of this section generalizes the connection made between the quantization of non-relativistic strings in flat space and the DLCQ description of flat space [3, 4]. We note that in the case considered in this paper the T-dual geometry is a different one from that of the parent theory, which we expect is a generic phenomena.

It would be interesting to analyze string theory in this time dependent pp-wave, which can be studied by fixing the light-cone gauge.

7. Comments

In this paper we have derived the worldsheet theory describing a sector of string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$; the non-relativistic sector. In brief, we have shown that once all the gauge symmetries are fixed, that the worldsheet theory describing this sector becomes a supersymmetric theory of free bosons and fermions propagating in an AdS_2 background. This opens the prospect of studying [7] the spectrum and interactions of this worldsheet theory to probe holography in a new stringy regime.

One crucial aspect in isolating non-relativistic strings was to consider a metric on AdS_5 that made manifest a space-like isometry, along which the non-relativistic strings could wind. In these coordinates, however, the metric is not static. Furthermore, the circle that the non-relativistic strings wind collapses both in the past and in the future.[18] One should be able to follow the time evolution of the wound strings using the explicit worldsheet theory that we have found. It would be interesting to analyze whether the T-dual, time dependent pp-wave can shed light on this. This set-up could provide an interesting setting in which to study time dependence in string theory using the holographic dual description.

[18] At $X^0 = \pm\frac{\pi}{2}R$ in the coordinates of (3.3).

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It is of great interest to fully quantize the worldsheet action and obtain the physics of both non-relativistic closed strings as well as that of open strings on D-branes. The reason that we can consider both types of strings is that we have performed a local worldsheet analysis, without committing to specific boundary conditions on the worldsheet. There are two types of D-branes that can be considered: longitudinal and transverse to the B-field. Quantizing open strings on a longitudinal D-brane leads to an AdS version of non-commutative open string theory (NCOS) [9, 10]. In order to get a non-trivial open string spectrum, there is no need of a spatial circle, which is required to obtain closed non-relativistic strings. The transverse D-branes have a non-trivial spectrum only if the open strings can wind and have a non-relativistic spectrum [12], just like the closed strings. Quantizing the model to obtain the spectrum of open and closed strings requires understanding the proper boundary conditions of the worldsheet fields on AdS_2 [7].

It is natural — and very interesting — to inquire what is the dual field theory realization of the non-relativistic string sector. In this section we make some preliminary remarks about the possible field theory interpretation.

The first thing that one must do to understand the field theory description of this sector is to identify the conformal boundary of (3.3). It is straightforward to show that the metric in the boundary is that of $\mathrm{AdS}_2\times\mathrm{S}^2$. It is given by:

$$ds^2 = -(dX^0)^2 + \cos^2\left(\frac{X^0}{R}\right)(dX^1)^2 + d\Omega_2\,. \tag{7.1}$$

The first observation is that this metric is conformally flat and is, therefore, a consistent background geometry on which to study $\mathcal{N}=4$ SYM.

Therefore, the physics of non-relativistic strings in $\mathrm{AdS}_5\times\mathrm{S}^5$ is captured by $\mathcal{N}=4$ SYM on $\mathrm{AdS}_2\times\mathrm{S}^2$, but with a time dependent metric. To further isolate the physics of non-relativistic strings, we must take the limit in (3.6). From the point of view of the field theory on the boundary, this has the effect of shrinking the size of the S^2 relative to AdS_2. Therefore, the $\mathcal{N}=4$ theory effectively reduces to a 1+1 dimensional field theory living in AdS_2. It is this reduction of $\mathcal{N}=4$ SYM that encodes the physics of non-relativistic strings in $\mathrm{AdS}_5\times\mathrm{S}^5$. It would be interesting to understand the integrability structure of this theory and its relation with integrability on the worldsheet.

In recent years, we have learned that in some situations, the dual gauge reproduces the worldsheet theory in the appropriate sector. In our case there is a tantalizing connection between the AdS_2 field theory living on the worldsheet and the 1+1 dimensional field theory that arises by taking the corresponding limit of $\mathcal{N}=4$ SYM. It would be very interesting to understand this relation in more detail [7].

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A. Invariant one forms

The superstring action in the $\mathrm{AdS}_5\times\mathrm{S}^5$ background is formulated using the coset superspace $\mathrm{SU}(2,2|4)/((\mathrm{SO}(4,1)\times\mathrm{SO}(5))$ [8]. We will mainly follow the notation used in that paper. The vector index for AdS_5 is $m=0,1,2,3,4$ and that for S^5 is $m'=1',2',3',4',5'$ with the flat metric $\eta_{mn}=diag(-;++++)$ and $\delta_{m'n'}=diag(+++++)$. The 10D gamma matrices are $\Gamma^m=\gamma^m\otimes 1\otimes\sigma_1$ for the AdS_5 part and $\Gamma^{m'}=1\otimes\gamma^{m'}\otimes\sigma_2$ for S^5. The charge conjugation matrix is $\mathcal{C}=CC'(i\sigma_2)$ and $\bar\theta\equiv\theta^T\mathcal{C}$. The chirality matrix is defined by $\Gamma_0\ldots\Gamma_4\Gamma_{1'}\ldots\Gamma_{5'}=\sigma_3$ and θ has the plus chirality $\sigma_3\theta=\theta$. The AdS_5 γ-matrix indices are $\alpha=1,2,3,4,$ and $\alpha'=1,2,3,4$ for S^5. $A=1,2$ are the indices of the $\sigma-$matrices and are often abbreviated. $I=1,2$ are the $\mathrm{SL}(2,\mathbb{R})$ indices on which the $\tau-$matrices act.

Given the coset parametrization in (3.1)

$$g=e^{iP_1X^1}e^{iP_0X^0}e^{iP_aX^a}e^{iP_{m'}X^{m'}}e^{iQ_{\alpha\alpha'I}\theta^{\alpha\alpha'I}}\,, \tag{A.1}$$

we can readily compute $\Omega=-ig^{-1}dg$ (2.3) using the $\mathrm{SU}(2,2|4)$ algebra. The forms associated with the bosonic generators of $\mathrm{SU}(2,2|4)$ are given by

$$\begin{aligned}\mathbf{L}^m&=e^m+\overline{\theta}\tilde\Gamma^{m\sharp}C_UD\theta\,, &\qquad \mathbf{L}^{mn}&=w^{mn}+\frac{1}{R}\overline{\theta}\tilde\Gamma^{mn}C_UD\theta\,,\\ \mathbf{L}^{m'}&=e^{m'}+\overline{\theta}\tilde\Gamma^{m'\sharp'}C_UD\theta\,, &\qquad \mathbf{L}^{m'n'}&=w^{m'n'}+\frac{1}{R}\overline{\theta}\tilde\Gamma^{m'n'}C_UD\theta\,,\end{aligned} \tag{A.2}$$

while the form associated to the supersymmetry generators is

$$\mathbf{L}=S_UD\theta\,. \tag{A.3}$$

Here we also use $\mathrm{SO}(4,2)$ and $\mathrm{SO}(6)$ Γ matrix tensors $\Gamma_{MN},\tilde\Gamma^{MN},(M=0,1,2,3,4,\sharp)$ and $\Gamma_{M'N'},\tilde\Gamma^{M'N'},(M'=1',2',3',4',5',\sharp')$ defined by

$$\begin{aligned}&\Gamma_{mn}=\gamma_{mn}\,, &&\Gamma_{m\sharp}=\gamma_m\,\tau_2 &&\tilde\Gamma^{mn}=\gamma^{mn}\,\tau_2\,\sigma_2\,, &&\tilde\Gamma^{m\sharp}=\gamma^m(-\sigma_2)\,,\\ &\Gamma_{m'n'}=\gamma_{m'n'}\,, &&\Gamma_{m'\sharp'}=\gamma_{m'}(i\tau_2)\,, &&\tilde\Gamma^{m'n'}=\gamma^{m'n'}(-\tau_2\,\sigma_2)\,, &&\tilde\Gamma^{m'\sharp'}=\gamma^{m'}(-i\sigma_2)\,.\end{aligned} \tag{A.4}$$

$e^{m,m'}$ are the bosonic vielbeins that determine the $\mathrm{AdS}_5\times$ S^5 metric and $w^{\cdot\cdot}$ are the usual bosonic spin connections associated with the stability group of the coset. $D\theta$ is the covariant derivative

$$D\theta=d\theta+\frac{1}{4}w^{mn}\,\Gamma_{mn}\theta+\frac{1}{2R}e^m\,\Gamma_{m\sharp}\theta+\frac{1}{4}w^{m'n'}\,\Gamma_{m'n'}\theta+\frac{1}{2R}e^{m'}\,\Gamma_{m'\sharp'}\theta \tag{A.5}$$

satisfying $D^2\theta=0$ and C_U and S_U are matrices defined by

$$S_U=\frac{\sinh U}{U}\,,\qquad C_U=2\frac{\cosh U-1}{U^2}\,, \tag{A.6}$$

$$\begin{aligned}U^2\equiv\frac{1}{R}\bigg(&\frac{1}{2}(\Gamma_{mn}\theta)(\overline{\theta}\tilde\Gamma^{mn})+(\Gamma_{m\sharp}\theta)(\overline{\theta}\tilde\Gamma^{m\sharp})+\\ &+\frac{1}{2}(\Gamma_{m'n'}\theta)(\overline{\theta}\tilde\Gamma^{m'n'})+(\Gamma_{m'\sharp'}\theta)(\overline{\theta}\tilde\Gamma^{m'\sharp'})\bigg)\,,\end{aligned} \tag{A.7}$$

where R is the radius of AdS_5 and S^5.

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The expressions for e^m and w^{mn} are given by ($\rho = \sqrt{X^a X_a}/R$),

$$e^0 = dX^0 \cosh\rho\,, \qquad e^1 = dX^1 \cosh\rho \cos\frac{X^0}{R}\,, \tag{A.8}$$

$$e^a = dX^a + dX^b(\eta_b{}^a - \frac{X_b X^a}{\rho^2 R^2})\left(\frac{\sinh\rho}{\rho} - 1\right), \tag{A.9}$$

$$w^{01} = -\frac{dX^1}{R}\sin\frac{X^0}{R}\,, \tag{A.10}$$

$$w^{0a} = \frac{X^a dX^0}{\rho R^2}\sinh\rho\,, \qquad w^{1a} = \frac{X^a dX^1}{\rho R^2}\sinh\rho\cos\frac{X^0}{R}\,, \tag{A.11}$$

$$w^{ab} = \frac{dX^a X^b - dX^b X^a}{\rho^2 R^2}(\cosh\rho - 1)\,. \tag{A.12}$$

while $e^{m'}$ and $w^{m'n'}$ are given by ($r = \sqrt{X^{m'} X_{m'}}/R$),

$$e^{m'} = dX^{m'} + dX^{n'}\left(\eta_{n'}{}^{m'} - \frac{X_{n'} X^{m'}}{r^2 R^2}\right)\left(\frac{\sin r}{r} - 1\right), \tag{A.13}$$

$$w^{m'n'} = \frac{dX^{m'} X^{n'} - dX^{n'} X^{m'}}{r^2 R^2}(\cos r - 1)\,. \tag{A.14}$$

B. Embedding of AdS

The AdS_5 geometry is embedded in a flat 6 dimensional space with the SO(4, 2) invariant metric $\eta_{MN} = (-;++++;-)$

$$ds^2 = \eta_{MN} du^M du^N\,. \tag{B.1}$$

It is a hyperboloid satisfying:

$$\eta_{NM} u^M u^N = -u^{0\,2} + u^{1\,2} + \left(\sum_{a=2}^{4} u^{a\,2}\right) - u^{\sharp\,2} = -R^2\,. \tag{B.2}$$

For the parametrization of the coset (3.1)

$$g = e^{iP_1 X^1} e^{iP_0 X^0} e^{iP_a X^a}\,, \tag{B.3}$$

the left invariant one forms are given in (A.8)–(A.12). The relation between the 6D coordinates u^M and the 5D coordinates X^m is:

$$u^M = \begin{pmatrix} u^0 \\ u^1 \\ u^a \\ u^5 \end{pmatrix} = R \begin{pmatrix} \cosh\rho \sin\frac{X^0}{R} \\ \cosh\rho\cos\frac{X^0}{R}\sinh\frac{X^1}{R} \\ \sinh\rho\,\frac{X^a}{R\rho} \\ \cosh\rho\cos\frac{X^0}{R}\cosh\frac{X^1}{R} \end{pmatrix}. \tag{B.4}$$

The parametrization of (B.4) is not global as the AdS_2 part of the coset parametrization (B.3), $e^{iP_1X^1}e^{iP_0X^0}$, is not a global one. The conformal boundary is $\mathrm{AdS}_2{\times}\mathrm{S}^2$, where the dual field theory lives.

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In the NR limit (3.6) we use $X^\mu = \omega x^\mu$, $R \to \omega R_0$. In this limit the AdS$_5$ hyperboloid becomes

$$u^M \to \omega R_0 \begin{pmatrix} \sin\frac{x^0}{R_0} \\ \cos\frac{x^0}{R_0}\sinh\frac{x^1}{R_0} \\ \frac{1}{\omega}\,\frac{x^a}{R_0} \\ \cos\frac{x^0}{R_0}\cosh\frac{x^1}{R_0} \end{pmatrix} \to \omega R_0 \begin{pmatrix} \sin\frac{x^0}{R_0} \\ \cos\frac{x^0}{R_0}\sinh\frac{x^1}{R_0} \\ \mathbf{0} \\ \cos\frac{x^0}{R_0}\cosh\frac{x^1}{R_0} \end{pmatrix}. \tag{B.5}$$

Using (B.2) with the renormalized coordinates $\hat{u}^M = \frac{u^M}{\omega}$, (B.5) becomes the parametrization of AdS$_2$ with radius R_0.

Although these are not global coordinates, there is an almost everywhere space-like Killing vector $V = \partial_{x^1}$. In these coordinates we can make the following identification

$$x^1 \simeq x^1 + L\,. \tag{B.6}$$

The metric of the compactified space is singular at the time boundaries of the chart $x^0 = \pm\frac{\pi}{2}R_0$.

C. Non-relativistic limit

In the non-relativistic limit (3.6) the coordinates are scaled as

$$X^\mu = \omega x^\mu\,, \qquad \theta = \sqrt{\omega}\,\theta_- + \frac{1}{\sqrt{\omega}}\,\theta_+\,, \qquad B_{\mu\nu} = \epsilon_{\mu\nu}\,, \qquad R = \omega R_0\,. \tag{C.1}$$

$\theta_\pm$ are defined using the projection operators $\mathbf{P}_\pm$,

$$\mathbf{P}_\pm\theta_\pm = \theta_\pm\,, \qquad \mathbf{P}_\pm = \frac{1}{2}(1 \pm \Gamma_*)\,, \qquad \Gamma_* \equiv \Gamma_0\Gamma_1\tau_3\,. \tag{C.2}$$

In this scaling, (in the following, we only write terms with negative powers of ω which are relevant in the limit $\omega \to \infty$)

$$e^0 = \omega\left(1 + \frac{\hat{\rho}^2}{2\omega^2}\right)dx^0\,, \qquad e^1 = \omega\left(1 + \frac{\hat{\rho}^2}{2\omega^2}\right)dx^1\cos\frac{x^0}{R_0}\,, \qquad e^a = dx^a\,. \tag{C.3}$$

Since the leading term in U^2 in (A.7) is of order ω^0, the trigonometric factors C_U and S_U in (A.6) are an infinite series at the order ω^0 (which is actually truncated due to the fermions). However in the κ-symmetry gauge choice $\theta_- = 0$ in (E.11) the leading terms in U^2 vanish and

$$\mathbf{P}_\pm C_U \mathbf{P}_\pm = 1 + \frac{1}{\omega^2}(C_U)^{(-2)}_{\pm\pm}\,, \qquad \mathbf{P}_\pm C_U \mathbf{P}_\mp = \frac{1}{\omega}(C_U)^{(-1)}_{\pm\mp} \tag{C.4}$$

and the same for S_U. In this gauge it is also shown that the leading $\omega^{\frac{1}{2}}$ term of $D\theta$ disappears and starts with a $\omega^{-\frac{1}{2}}$ term

$$\mathbf{P}_+(D\theta) = \frac{1}{\sqrt{\omega}}\mathbf{D}\theta_+\,, \qquad \mathbf{P}_-(D\theta) = O\left(\frac{1}{\omega^{3/2}}\right), \tag{C.5}$$

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where

$$\mathbf{D}\theta_+ = \left(d + \frac{1}{2R_0}\mathbf{v}^\mu \gamma_\mu \tau_2 + \frac{1}{4}\mathbf{w}^{\mu\nu}\gamma_{\mu\nu}\right)\theta_+ \,, \tag{C.6}$$

$$\mathbf{v}^\mu = \left(dx^0, dx^1 \cos\frac{x^0}{R_0}\right), \qquad \mathbf{w}^{01} = w^{01} = -\frac{dx^1}{R_0}\sin\frac{x^0}{R_0}\,. \tag{C.7}$$

These expressions all simplify drastically in the $\omega \to 0$ limit. The Cartan one forms (A.2) in the non-relativistic limit in the $\theta_- = 0$ gauge (E.11) are

$$\begin{aligned}
\mathbf{L}^\mu &= \omega \mathbf{L}^{\mu(1)} + \frac{1}{\omega}\mathbf{L}^{\mu(-1)} + \cdots = \omega\,\mathbf{v}^\mu + \frac{1}{\omega}\left(\frac{\hat\rho^2}{2}\mathbf{v}^\mu + \overline{\theta}_+\tilde\Gamma^{\mu\sharp}\mathbf{D}\theta_+\right) + \cdots \\
\mathbf{L}^a &= \mathbf{L}^{a(0)} + \cdots = dx^a + \cdots \\
\mathbf{L}^{m'} &= \mathbf{L}^{m'(0)} + \cdots = dx^{m'} + \cdots \\
\mathbf{P}_+\mathbf{L}^{\alpha\alpha' I} &= \frac{1}{\sqrt{\omega}}\mathbf{L}_+^{\alpha\alpha' I(-1/2)} + \cdots = \frac{1}{\sqrt{\omega}}(\mathbf{D}\theta_+)^{\alpha\alpha' I} + \cdots\,, \\
\mathbf{P}_-\mathbf{L}^{\alpha\alpha' I} &= \frac{1}{\omega^{3/2}}\mathbf{L}_-^{\alpha\alpha' I(-3/2)} + \cdots = \cdots\,.
\end{aligned} \tag{C.8}$$

where $\cdots$ are terms that will not contribute to the action in the $\omega \to \infty$ limit.

D. Divergent terms of the lagrangian in the Nambu-Goto form

Here, we derive the crucial identity

$$d(\omega^2(\det \mathbf{L}^{\mu(1)})_{\text{fermionic}} + \mathcal{L}^{(WZ)}_{\text{div}}) = 0 \tag{D.1}$$

used in the main text (4.8). Since $\det \mathbf{L}^{\mu(1)} = \det \mathbf{e}^{\mu(1)} + (\det \mathbf{L}^{\mu(1)})_{\text{fermionic}}$ and $\det \mathbf{e}^{\mu(1)}$ is trivially closed (it is a top form), (4.8) is equivalent to

$$d(\omega^2 \det \mathbf{L}^{\mu(1)} + \mathcal{L}^{(WZ)}_{\text{div}}) = 0\,. \tag{D.2}$$

Noting that $\epsilon^{01} = -\epsilon_{01} = 1$, we have

$$\det(\mathbf{L}_i^\mu) d^2\xi = -\frac{1}{2}\epsilon_{\mu\nu}\mathbf{L}^\mu \mathbf{L}^\nu\,. \tag{D.3}$$

Multiplying by d and using the Maurer-Cartan equation (2.4) we get

$$d[\det(\mathbf{L}_i^\mu)d^2\xi] = -\epsilon_{\mu\nu}(d\mathbf{L}^\mu)\mathbf{L}^\nu = -\epsilon_{\mu\nu}(-\mathbf{L}^{\mu\rho}\mathbf{L}_\rho - \mathbf{L}^{\mu a}\mathbf{L}_a + \bar{\mathbf{L}}\tilde\Gamma^{\mu\sharp}\mathbf{L})\mathbf{L}^\nu\,. \tag{D.4}$$

The ω^2 term in $d\mathcal{L}^{(NG)}$ comes from the last term and satisfies

$$[d[-T\sqrt{-\det g}\, d^2\xi]]_{\omega^2\,\text{term}} = T\epsilon_{\mu\nu}(\bar{\mathbf{L}}_-^{(\frac{1}{2})}\tilde\Gamma^{\mu\sharp}\mathbf{L}_-^{(\frac{1}{2})})\mathbf{L}^{\nu(1)}\,. \tag{D.5}$$

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Using an identity which holds from the definition of θ_-

$$\epsilon_{\mu\nu}\tilde{\Gamma}^{\mu\sharp}\mathbf{P}_- = \epsilon_{\mu\nu}\gamma^{\mu}(-\sigma_2)\mathbf{P}_- = -i\Gamma_{\nu}\sigma_3\tau_3\mathbf{P}_- \tag{D.6}$$

the divergent piece of the Nambu-Goto term can be written as

$$[d[-T\sqrt{-\det g}\, d^2\xi]]_{\omega^2\text{ term}} = T(\bar{\mathbf{L}}_-^{(\frac{1}{2})}(-i\Gamma_{\nu}\tau_3)\mathbf{L}_-^{(\frac{1}{2})})\mathbf{L}^{\nu(1)}\,. \tag{D.7}$$

On the other hand the WZ action is given by (2.5)

$$d\mathcal{L}^{(WZ)} = -iT\,\bar{\mathbf{L}}(\Gamma_m\mathbf{L}^m + \Gamma_{m'}\mathbf{L}^{m'})\tau_3\mathbf{L}\,. \tag{D.8}$$

The ω^2 term comes from $m = \mu$ and by isolating the $\mathbf{L}_-$ term as

$$[d\mathcal{L}^{(WZ)}]_{\omega^2\text{ term}} = -iT\,(\bar{\mathbf{L}}_-^{(\frac{1}{2})}\Gamma_{\mu}\mathbf{L}^{\mu(1)})\tau_3\,\mathbf{L}_-^{(\frac{1}{2})} \tag{D.9}$$

and it cancels with (D.7), which implies (D.2). Therefore, the ω^2 piece of the non-relativistic Nambu-Goto lagrangian is a total derivative.

E. Nambu-Goto form of the action and κ-symmetry

By integrating out the auxiliary metric h_{ij} in the Polyakov action (2.5), we get the Nambu-Goto action:

$$\mathcal{L} = -T\,\mathcal{L}^{(NG)} - T\,\mathcal{L}^{(WZ)}\,. \tag{E.1}$$

The first term is the Nambu-Goto Kinetic term (2.6)

$$\mathcal{L}^{(NG)} = \sqrt{-\det G_{ij}}\,, \tag{E.2}$$

with the induced metric

$$G_{ij} = \mathbf{L}_i^m\mathbf{L}_j^n\eta_{mn} + \mathbf{L}_i^{m'}\mathbf{L}_j^{n'}\delta_{m'n'}\,. \tag{E.3}$$

The second contribution is the WZ lagrangian whose exterior derivative is given as an invariant three form

$$d(\mathcal{L}^{(WZ)}d^2\xi) = i\,\bar{\mathbf{L}}(\Gamma_m\mathbf{L}^m + \Gamma_{m'}\mathbf{L}^{m'})\tau_3\mathbf{L}\,. \tag{E.4}$$

$\mathbf{L}^m, \mathbf{L}^{m'}$ and $\mathbf{L}^{\alpha\alpha' AI}$ are left invariant one forms constructed from the coset $\frac{\mathrm{SU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$, see appendix A.

This relativistic action is invariant under diffeomorphism and κ-symmetry [8]. The κ-symmetry transformations are

$$[\delta\theta] = \frac{1}{2}(1-\Gamma_{\kappa})\kappa \qquad \text{and} \qquad [\delta X]^m = [\delta X]^{m'} = 0\,, \tag{E.5}$$

where $[\delta\theta]^{\alpha\alpha' I}, [\delta X]^m$ and $[\delta X]^{m'}$ are $\mathbf{L}^{\alpha\alpha' I}, \mathbf{L}^m$ and $\mathbf{L}^{m'}$ in which dX^A is replaced by δX^A for the superspace coordinates $X^A = (X^m, X^{m'}, \theta)$.[19] Γ_{κ} satisfies $\Gamma_{\kappa}^2 = 1$ and is given by

$$\Gamma_{\kappa} \equiv \frac{1}{2\sqrt{-G}}\epsilon^{ij}\not{L}_i\not{L}_j\tau_3\,, \qquad \not{L} \equiv \Gamma_m\mathbf{L}^m + \Gamma_{m'}\mathbf{L}^{m'}\,, \tag{E.6}$$

where G is the determinant of the induced metric. The κ-symmetry transformations δX^A of the supercoordinates are determined from (E.5).

[19] In the Polyakov form of the lagrangian the metric h_{ij} is also transformed properly [8].

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Let us know find the κ-transformation for the scaled variables $\theta_\pm$ as an expansion in terms of ω. In order to do that we rescale the parameter κ

$$\kappa = \sqrt{\omega}\kappa_- + \frac{1}{\sqrt{\omega}}\kappa_+ \tag{E.7}$$

and we do the power expansion of Γ_κ in powers of ω

$$\Gamma_\kappa = \Gamma_* + \frac{\epsilon^{ij}}{\omega\sqrt{-g}}(\Gamma_\mu \mathbf{L}_i^{\mu(1)})(\Gamma_b \mathbf{L}_j^{b(0)} + \Gamma_{m'} \mathbf{L}_j^{m'(0)})\tau_3 + O(\omega^{-2}) \tag{E.8}$$

where $g = \det(\eta_{\mu\nu}\mathbf{L}_i^{\mu(1)}\mathbf{L}_j^{\nu(1)})$. We have

$$[\delta\theta]_- = \kappa_- \,, \qquad [\delta\theta]_+ = -\frac{\epsilon^{ij}}{2\sqrt{-g}}(\Gamma_\mu \mathbf{L}_i^{\mu(1)})(\Gamma_b \mathbf{L}_j^{b(0)} + \Gamma_{m'} \mathbf{L}_j^{m'(0)})\tau_3\kappa_- + \cdots \,. \tag{E.9}$$

Notice that κ_+ does not contribute the lowest order in ω. From (E.9) we can see that θ_- could be gauged away using the κ-transformation since

$$\delta\theta_-|_{\theta_-=0} = \kappa_- \,. \tag{E.10}$$

Therefore, θ_- is the gauge degree of freedom associated to κ-transformation. In this paper we choose the κ-symmetry gauge condition:

$$\theta_- = 0 \,. \tag{E.11}$$

This gauge is also valid for the Polyakov action. Especially since $\theta_- = 0$ is stable under the diffeomorphism and the Weyl symmetry, the choice is used in the conformal gauge of the metric.

We thus see that the non-relativistic limit guarantees that the non-relativistic string action inherits the gauge symmetries of the parent theory. The symmetries of the non-relativistic lagrangian are a consequence of the symmetries of the parent relativistic action and the fact that the divergent term of the non-relativistic expansion of the relativistic action is total derivative, as we proved in appendix D.

F. Non-relativistic string contraction of SU(2, 2|4) algebra

The space-time supersymmetry algebra of the non-relativistic action (4.17) (4.20) is given by a supersymmetrization of the Newton-Hooke group. It can be obtained by the non-relativistic contraction of $\mathrm{SU}(2,2|4)$. Let us study this algebra using SO(4,2)×SO(6) notations. Letting $\hat{M} = (M, M')$ run over SO(4,2) and SO(6) indices the algebra is

$$\begin{aligned}
\left[M_{\hat{M}\hat{N}}, M_{\hat{R}\hat{S}}\right] &= -i\eta_{\hat{N}[\hat{R}}M_{\hat{M}\hat{S}]} + i\eta_{\hat{M}[\hat{R}}M_{\hat{N}\hat{S}]} \,, && \text{(F.1)}\\
\left[Q_\pm,\ M_{\hat{M}\hat{N}}\right] &= \frac{i}{2}Q_\pm(\Gamma_{\hat{M}\hat{N}}) && \text{(F.2)}\\
\{Q_{\alpha\alpha' I},\ Q_{\beta\beta' J}\} &= -\frac{i}{R}(\mathcal{C}\tilde{\Gamma}^{\hat{M}\hat{N}})_{\alpha\alpha' I,\beta\beta' J}\ M_{\hat{M}\hat{N}} \,, && \text{(F.3)}
\end{aligned}$$

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where $P_m = \frac{1}{R} M_{m\sharp}$, $P_{m'} = \frac{1}{R} M_{m'\sharp'}$. Corresponding to the rescaling (3.6) we rescale the generators as

$$P_\mu \to \frac{1}{\omega} P_\mu\,, \qquad M_{\mu a} \to \omega B_{\mu a}\,, \qquad Q_- \to \frac{1}{\sqrt{\omega}} Q_-\,, \qquad Q_+ \to \sqrt{\omega} Q_+\,, \tag{F.4}$$

where the supersymmetry generators $Q_\pm$ are defined using the projection operator $\mathbf{P}_\pm = \frac{1}{2}(1 \pm \Gamma_0 \Gamma_1 \tau_3)$. The non-zero bosonic AdS_5 commutators in the $\omega \to \infty$ limit become

$$\begin{aligned}
[P_\mu,\ M_{\nu\rho}] &= -i\eta_{\mu[\nu}\ P_{\rho]}\,, \qquad [M_{\mu\nu}, B_{\rho b}] = -i\eta_{[\nu\rho} B_{\mu]b}\,,\\
[P_a, M_{cd}] &= -i\eta_{a[c}\ P_{d]}\,, \qquad [B_{\mu a}, M_{cd}] = -i\eta_{a[c} B_{\mu d]}\,,\\
[M_{ab}, M_{cd}] &= -i(\eta_{b[c} M_{ad]} - \eta_{a[c} M_{bd]})
\end{aligned} \tag{F.5}$$

and

$$\begin{aligned}
[P_\mu,\ P_\nu] &= -i\ (\frac{1}{R_0^2})\ M_{\mu\nu}\,, \qquad [P_\mu,\ P_b] = -i\ (\frac{1}{R_0^2})\ B_{\mu b}\,,\\
[P_\mu,\ B_{\nu b}] &= -i\ \eta_{\mu\nu}\ P_b\,.
\end{aligned} \tag{F.6}$$

The bosonic S^5 algebra becomes euclidean algebra in the limit,

$$\begin{aligned}
[P_{m'},\ M_{n'\ell'}] &= -i\eta_{m'[n'}\ P_{\ell']}\,,\\
[M_{m'n'}, M_{\ell' r'}] &= -i(\eta_{n'[\ell'} M_{m'r']} - \eta_{m'[\ell'} M_{n'r']})\,.
\end{aligned} \tag{F.7}$$

The QM algebra (F.2) becomes using

$$\begin{aligned}
\Gamma_{\mu\sharp}\mathbf{P}_\pm &= \mathbf{P}_\pm \Gamma_{\mu\sharp}\,, \quad \Gamma_{a\sharp}\mathbf{P}_\pm = \mathbf{P}_\mp \Gamma_{a\sharp}\,, \quad \Gamma_{\mu\nu}\mathbf{P}_\pm = \mathbf{P}_\pm \Gamma_{\mu\nu}\,, \quad \Gamma_{\mu a}\mathbf{P}_\pm = \mathbf{P}_\mp \Gamma_{\mu a}\,,\\
\Gamma_{ab}\mathbf{P}_\pm &= \mathbf{P}_\pm \Gamma_{ab}\,, \quad \Gamma_{m'\sharp'}\mathbf{P}_\pm = \mathbf{P}_\mp \Gamma_{m'\sharp'}\,, \quad \Gamma_{m'n'}\mathbf{P}_\pm = \mathbf{P}_\pm \Gamma_{m'n'}\,,
\end{aligned} \tag{F.8}$$

$$\begin{aligned}
[Q_\pm,\ P_\mu] &= \frac{i}{2R_0} Q_\pm(\Gamma_{\mu\sharp})\,, \qquad [Q_-,\ P_a] = \frac{i}{2R_0} Q_+(\Gamma_{a\sharp})\,,\\
[Q_\pm,\ M_{\mu\nu}] &= \frac{i}{2} Q_\pm(\Gamma_{\mu\nu})\,, \qquad [Q_-,\ P_{m'}] = \frac{i}{2R_0} Q_+(\Gamma_{m'\sharp'})\\
[Q_\pm,\ M_{ab}] &= \frac{i}{2} Q_\pm(\Gamma_{ab})\,, \qquad [Q_-,\ B_{\mu a}] = \frac{i}{2} Q_+(\Gamma_{\mu a})\,,\\
[Q_\pm,\ M_{m'n'}] &= \frac{i}{2} Q_\pm(\Gamma_{m'n'})\,.
\end{aligned} \tag{F.9}$$

The QQ algebra (F.3) becomes using

$$\begin{aligned}
&\mathbf{P}_\pm^T \mathcal{C} = \mathcal{C}\mathbf{P}_\mp\,,\\
&\tilde{\Gamma}^{\mu\sharp}\mathbf{P}_\pm = \mathbf{P}_\mp \tilde{\Gamma}^{\mu\sharp}\,, \quad \tilde{\Gamma}^{a\sharp}\mathbf{P}_\pm = \mathbf{P}_\pm \tilde{\Gamma}^{a\sharp}\,, \quad \tilde{\Gamma}^{\mu\nu}\mathbf{P}_\pm = \mathbf{P}_\mp \tilde{\Gamma}^{\mu\nu}\,, \quad \tilde{\Gamma}^{\mu a}\mathbf{P}_\pm = \mathbf{P}_\pm \tilde{\Gamma}^{\mu a}\,,\\
&\tilde{\Gamma}^{ab}\mathbf{P}_\pm = \mathbf{P}_\mp \tilde{\Gamma}^{ab}\,, \quad \tilde{\Gamma}^{m'\sharp'}\mathbf{P}_\pm = \mathbf{P}_\pm \tilde{\Gamma}^{m'\sharp'}\,, \quad \tilde{\Gamma}^{m'n'}\mathbf{P}_\pm = \mathbf{P}_\mp \tilde{\Gamma}^{m'n'}\,,
\end{aligned} \tag{F.10}$$

and taking $\omega \to \infty$ limit

$$\begin{aligned}
\{Q_-,\ Q_-\} &= -\frac{i}{R_0}[\mathcal{C}(2\tilde{\Gamma}^{\mu\sharp}\ R_0 P_\mu + \tilde{\Gamma}^{\mu\nu} M_{\mu\nu} + \tilde{\Gamma}^{ab} M_{ab} + \tilde{\Gamma}^{m'n'} M_{m'n'})\mathbf{P}_-]\,,\\
\{Q_+,\ Q_-\} &= -\frac{2i}{R_0}[\mathcal{C}(\tilde{\Gamma}^{a\sharp}\ R_0 P_a + \tilde{\Gamma}^{\mu a} B_{\mu a} + \tilde{\Gamma}^{m'\sharp'}\ R_0 P_{m'})\mathbf{P}_-]\,.
\end{aligned} \tag{F.11}$$

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